ASYMPTOTIC STABILITY FOR GAUSS METHODS FOR NEUTRAL DELAY DELAY DIFFERENTIAL EQUATIONS

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Abstract. In [4] we proved that all Gauss methods are $N\tau(0)$ -compatible for neutral delay differential equations (NDDEs) of the form :

$$\begin{aligned} &y'(t) = ay(t) + by(t-\tau) + cy'(t-\tau), \quad t > 0, \\ &y(t) = g(t), \qquad\qquad -\tau \le t \le 0, \end{aligned} \tag{0.1}$$

where a,b,c are real, $g(t)$ is a continuous real valued function. In this paper we are going to use the theory of order stars to characterize the asymptotic stability properties of Gauss methods for NDDEs. And then proved that all Gauss methods are $N\tau(0)$ -stable.

1. Introduction

In order to simplify the notation, without loosing the generality of the problem we can fix the delay equal to 1. For the sake of the simplicity, in the sequel we deal with the following test equation

$$\begin{aligned} &y'(t) = ay(t) + by(t-1) + cy'(t-1), \quad t > 0, \\ &y(t) = g(t), \qquad\qquad -1 \le t \le 0, \end{aligned} \tag{0.2}$$

where a,b,c are real, $\tau > 0$, $g(t)$ is a continuous real valued function. Its characteristic equation is given by :

$$\lambda - a - b\exp(-\lambda) - c\lambda\exp(-\lambda) = 0 \,. \tag{0.3}$$

It is known that the set of triplet (a, b, c) for which the solution $y(t)$ of (0.2) tend to zero when $t \to \infty$ is given by :

$$\Sigma_* = \{(a,b,c) \in \mathbb{R}^3 \quad \text{all root } \lambda \text{ of (1.2) satisfying } \operatorname{Re}[\lambda] < 0, |c| < 1\}.$$

And λ is uniformly bounded away from the imaginary axis.

It can be rewriten as $\Sigma_* = \Sigma \cup \mathrm{E}$ where

$$\mathrm{E} = \{(a,b,c) \in \mathbb{R}^3, \quad a + |b| < 0 \quad \text{and} \quad |c| < 1\},$$

$$\Sigma=\left\{(a,b,c)\in R^3,\ |a|<-b,\text{and}\sqrt{b^2-a^2}<\sqrt{1-c^2}\arccos\left(\frac{c+\rho}{1+c\rho}\right)\text{ with }|c|<1\right\}.$$

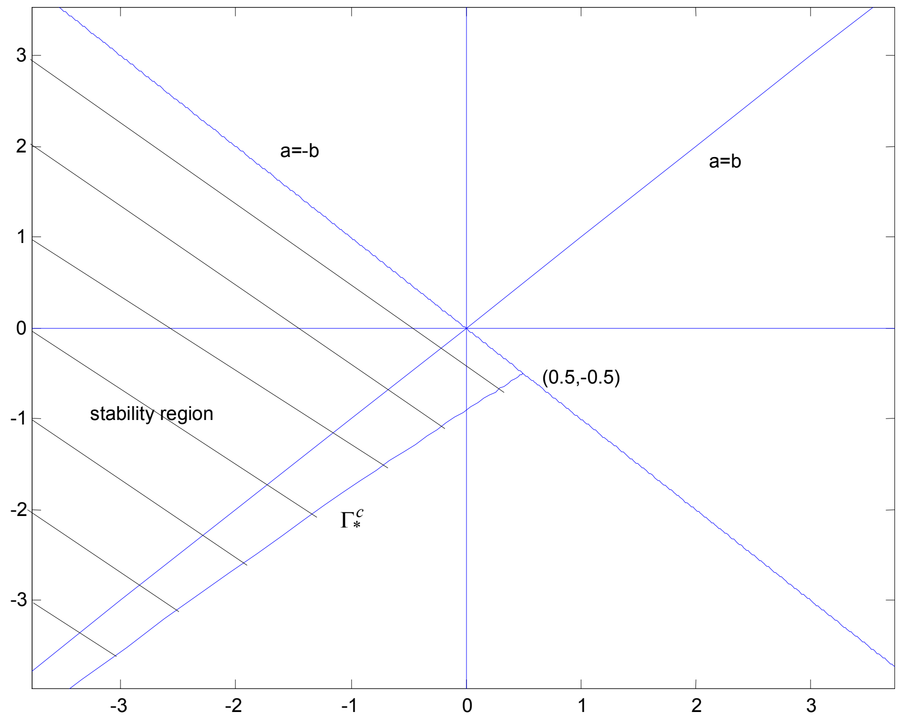
with $\rho=-\dfrac{a}{b}$. This set is bounded in the right by the plane

$P=\{(a,b,c)\in R^3, a=-b$ with $a<1-c$, $|c|<1\}$ and the transcendental surface

$\Gamma_*=\{(a_*(\theta,c),b_*(\theta,c),c)\in R^3 \mid \theta\in(0,\pi)$ and $a<1-c,|c|<1\}$,

with

$$a_*(\theta,c)=\frac{\theta\cos\theta-c\theta}{\sin\theta},\qquad b_*(\theta,c)=\frac{c\theta\cos\theta-\theta}{\sin\theta}.$$



1.1 Runge -Kutta methods for NDDEs

Let us consider the following s-stage RK method

$$y_{n+1}=y_n+h\sum_{i=1}^{v}w_iK_i^{n+1} \tag{1.3}$$

$$K_i^{n+1}=f\left(t_n+c_ih,\ y_n+h\sum_{j=1}^{v}a_{ij}K_j^{n+1},\ y_{n-m}+h\sum_{j=1}^{v}b_{ij}K_j^{n-m+1},\ \sum_{j=1}^{v}c_{ij}K_j^{n-m+1}\right),$$

i=1,2,…, s, where $h=\tau/m$, $c_i=\sum_{j=1}^{s}a_{ij}$ .Here $W=[w_1,...,w_s]^T$ and the matrix $A=[a_{ij}]_{i,j=1}^{s}$ define a RK method for ODEs. [3, 6, 13]. The second argument in f can be interpreted as an approximation to $y(t)$ at the intermediate point $t_n + c_j h$. Similarly the third argument in f can be interpreted as an approximation to $y(t_{n-m} + c_j h)$ and the fourth to $y'(t_{n-m} + c_j h)$ usually $b_{ij} = w_j(c_i)$ and $c_{ij} = w'_j(c_i)$ where $w_i(\theta), i = 1,...s$ are polynomials which define the

natural continuous extension of RK method, i.e. polynomials such that the approximate solution y_h define the whole interval of integration is given by

$$y_h(t_n + \theta h) = y_h(t_n) + h\sum_{i=1}^{s} w_i(\theta)K_i^{n+1}, \qquad n = 0,1,\Lambda, \qquad \theta \in (0,1].$$

Take $B = [b_{ij}]_{i,j=1}^{s}$, $C = [c_{ij}]_{i,j=1}^{s}$ and $e = [1,\Lambda,1]^T$, I is the identity matrix.

In our present work we consider constant step-size and equal to an integer sub-multiple of the delay i.e. $h = 1/m$, m is positive number. When apply (1.3) in the case *A=B, C=I* to the test equation (1.1), the following difference equation in some case is also useful for examining the stability c.f. Bellen *et* all [3]

$$P(\zeta) = 1 + (\frac{a}{m} + \frac{b}{m\zeta^m})W^T((1 - \frac{c}{\zeta^m})I - (\frac{a}{m} + \frac{b}{m\zeta^m})A)^{-1}e - \zeta = 0, \tag{1.4}$$

$$\zeta = R(z), \qquad z = \frac{a + b\zeta^{-m}}{m(1 - c\zeta^{-m})}, \tag{1.5}$$

with

$$R(z) = 1 + zW^T(I - zA)^{-1}e,$$

where $R(z)$ is the stability function of the method [6, 11] the numerical solution of (1.1) is asymptotically stable if and only if $|\zeta| < 1$ whenever ζ satisfies (1.5). Or we can present it as

$$\Sigma_m = \{(a,b,c) \in R^3, \text{ all root } \zeta \text{ of (1.5) satisfying } |\zeta| < 1\}. \tag{1.6}$$

Definition 2.2 The $N\tau(0)$-stability region of the numerical step-by-step method for NDDEs is the set

$$S_{N\tau(0)} = \mathrm{I}_{m\geq 1} \Sigma_m.$$

Definition 2.1 A Runge-Kutta method is said $N\tau(0)-$ stable if for a,b,c real, we have $\Sigma_* \subset \Sigma_m$ for all $m \geq 1$.

2 Stability analysis with respect to c

In order to show $\Sigma_* \subseteq \Sigma_m$ we apply the root locus technique [2, 5]. It is hard to represent geometrically the set of all $(a,b,c) \in R^3$ satisfying (1.5), so it is more convenient to reduce the dimension and get a 2-dimensional parametrization. We choose c as a parameter and develop the stability analysis in the (a, b)-plane. We define by :

$$\Sigma_*^c = \{(a,b) \in R^2, \text{ all root } \zeta \text{ of (1.2) satisfying } |\zeta| < 1\}.$$
$$\Sigma_m^c = \{(a,b) \in R^2, \text{ all root } \zeta \text{ of (1.5) satisfying } |\zeta| < 1\}.$$

So we need only to show $\Sigma_*^c \subseteq \Sigma_m^c$ for any fixed $c \in (-1,1)$. Since z of (1.5) depends continuously on a, b, ($R(z) = P(z)/Q(z)$) it's root of $(mz-a)P(z)^m - (mcz+b)Q(z)^m = 0$, also $\zeta = R(z)$ depends continuously on a, b Therefore it is sufficient to prove for the values of (a,b) for fixed c satisfying (1.5) with $|\zeta| = 1$ all lie outside in the analytical stability region Σ_*^c.

For $x = y = \phi = 0$ we get the line $a + b = 0$ (as the analytic stability region), if $\sin m\phi = 0$ and $y = 0$, it happens for $\phi = s\pi$, we get another line $a \pm b = mx(1 \pm c)$. For $0 < \phi < s\pi$ we get

$$a_m(\phi) = my\frac{\cos m\phi - c}{\sin m\phi} + mx, \qquad (2.2)$$
$$b_m(\phi) = -my\frac{1 - c\cos m\phi}{\sin m\phi} - cmx.$$

3. Stability analysis for Gauss-methods

We consider the curves of the locus roots (2.2) with $x = 0$ (Gauss methods) and denote by

$$\Gamma_k^c = \{(a_m(\phi,c), b_m(\phi,c),) \in R^2, m\phi \in ((k-1)\pi, k\pi)\},$$

with k a positive integer.

Proposition 4.1 *The curves Γ_k^c are all separated.*

If k is even the curve Γ_k^c lies in the sector $|a| < b$, if k is odd the curve Γ_k^c lies in the sector $|a| < -b$. And the curve Γ_1^c starts at the point $\delta_c = (1-c, c-1)$, which is the double bifurcation point

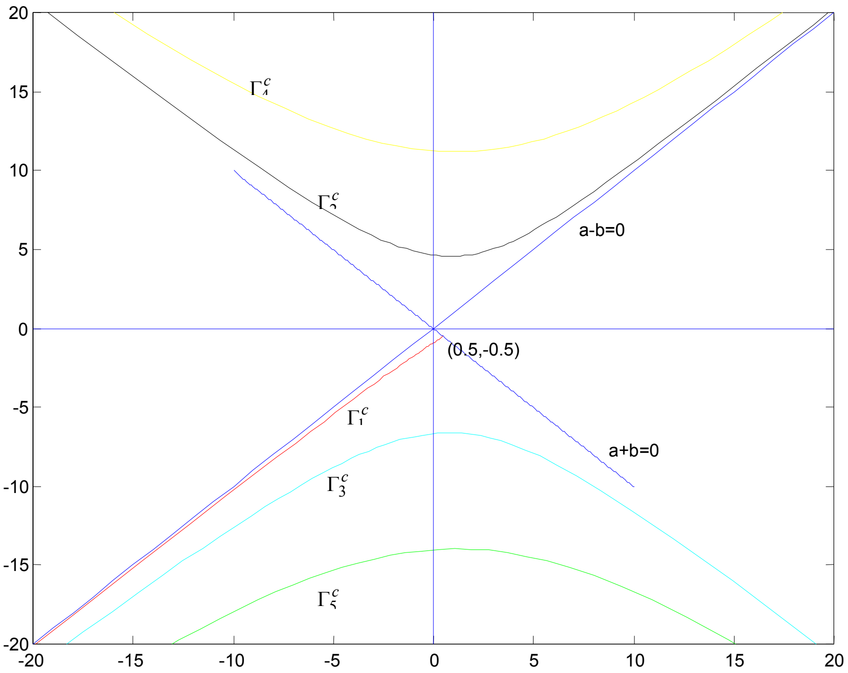


Figure 2. Plot of Γ_k^c for $c = 0.5,\ m = 10$ for implicit midpoint.

Theorem 4.1 *All Gauss-methods are $N\tau(0)-stable$.*

Theorem 2.2 For Gauss-methods the stability region Σ_m^c is the set which is bounded to the right by the half line $a+b=0$ with $a<1-c$ and by the curve Γ_1^c

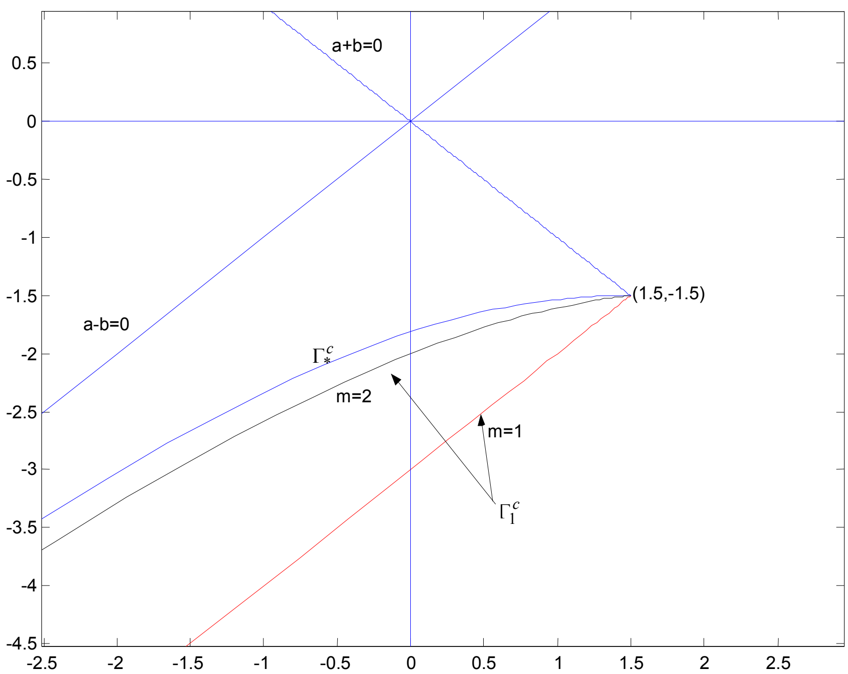


Figure 3 Stability region for implicit midpoint for $c = -0.5$.