Sur la nature asymptotique de la fonction de Hilbert attachée au cône isotrope d'une filtration noethérienne

D. Sangaré
Université d'Abobo - Adjamé, UFR- SFA,
Département de Mathématiques
22 BP 1709, Abidjan 22, Côte d'Ivoire
e-mail : dsangare@yahoo.fr

Résumé

De nombreuses fonctions numériques prennent les mêmes valeurs qu'un polynôme ou une suite finie de polynômes pour les valeurs suffisamment grandes de la variable entière. Ici on étudie cette question dans le cas particulier où l'on prend comme fonction numérique la fonction de Hilbert $n \longmapsto \varphi_f(n) = dim_k(\frac{I_n}{m\, I_n})$ attachée au cône isotrope $G(A, m, f) = \bigoplus_{n \geq 0} \frac{I_n}{m\, I_n}$ d'une filtration noethérienne $f = (I_n)$ d'un anneau local noethérien (A, m, k) , où $k = \frac{A}{m}$ est le corps des restes et où dim_k désigne la dimension de k- espace vectoriel.

Mots clés : Fonctions numériques, fonctions de Hilbert, filtrations, cône isotrope

Let (A, m, k) be a noetherian local ring with residue field $k = \frac{A}{m}$ and let I be an ideal of A. Then it is known from Northcott and Rees [NR] that the numerical function $n \longmapsto \varphi_f(n) = dim_k(\frac{I_n}{m\, I_n})$

$n \longmapsto \varphi_I(n) = dim_k(\frac{I^n}{m\, I^n})$ which denotes the dimension of the k- vector space $\frac{I^n}{m\, I^n}$ is of polynomial type. The analytic spread of the ideal I was defined by the above authors as the integer $\lambda(I) = 1 + deg\varphi_I$, where $deg\varphi_I$ is the degree of the polynomial associated with the numerical function φ_I. Extensions of this concept to filtrations had been investigated in particular by D. Sangare[a] in previous papers. Unfortunately if $f = (I_n)$ is a filtration on a noetherian local ring (A, m, k) then the numerical function $n \longmapsto \varphi_f(n) = dim_k(\frac{I_n}{m\, I_n})$ need not be of polynomial type even for nice filtrations like noetherian filtrations. Here we show that φ_f is of polynomial type in each of the following cases :

(i) f is $I-$ good

(ii) f is strongly noetherian

In addition, we prove that for an $I-$ good filtration $f = (I_n)$ on a noetherian local ring (A, m, k), the analytic spread of f which is defined as being the integer $\lambda(f) = 1 + deg\varphi_f$, where deg φ_f is the degree of the polynomial associated with the function φ_f , is equal to the analytic spread of the ideal I, hence it does not depend on the I- good filtration f.

1 Introduction

Throughout this paper, (A, m, k) will denote a given noetherian local ring with residue field $k = \frac{A}{m}$ and I a proper ideal of A.

Definition 1

Let $k \geq 1$ and $d \geq -1$ be integers and let $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a numerical function.

(i) φ is said to be of polynomial type of degree d if there exists a polynomial $P \in \mathbb{Q}[X]$ of degree d such that $\varphi(n) = P(n) \forall n \gg 0$, which means "for all large n".

(ii) φ is called a quasi- polynomial function of period k and degree d if there exists a sequence $F = (F_0,\ F_1,\ ...,\ F_{k-1})$ of polynomials $F_j \in \mathbb{Q}[X]$ such that for $j = 0, 1, ..., k-1$,

(i) $\varphi(n) = F_j(n) \forall n \gg 0$ with $n \equiv j (mod k)$ and

(ii) $Max_{0 \leq\ j \leq\ k-1} \deg F_j = d$.

F is called the quasi-polynomial associated with φ and d is called the degree of F.

Here the zero polynomial has degree -1.

Let (A, m, k) be a noetherian local ring and let I be an ideal of A, where $k = \frac{A}{m}$ is its residue field. Then the numerical function $n \longmapsto \varphi_I\ (n) = dim_k(\frac{I^n}{m\ I^n})$ which is the dimension of the k - vector space $\frac{I^n}{m\ I^n}$ is of polynomial type, see [NR]. The analytic spread $\lambda(I)$ of the ideal I was defined by the above authors as :

(1) $\lambda(I) = 1 + deg\varphi_I$, where $deg\varphi_I$ is the degree of the polynomial associated with the numerical function φ_I.

This concept has been intensively studied since its introduction. Its success comes first of all from the various ways it can be interpreted. Let us recall some of them :

Suppose k is infinite. Then following [NR] :

(2) $\lambda(I) = \mu(J)$, where J is a minimal reduction of I and $\mu(J)$ the cardinal of a minimal generating set of J and

(3) $\lambda(I) = max\{r, \exists a_1,\ a_2,\ ...,\ a_r \in I$ which are analytically independent in $I\}$.

Now without any condition on the cardinal of k, we have :

(4) $\lambda(I) = dim G(A, m, I) = dim \frac{R(A,\ I)}{m\ R(A,\ I)} = dim \frac{\mathcal{R}(\text{A, I})}{(u, m)\ \mathcal{R}(\text{A, I})}$, where

$G(A, m, I) = \bigoplus_{n \geq 0} \frac{I^n}{m\ I^n}$ is the fiber cone of I and $R(A, I) = \bigoplus_{n \geq 0} I^n X^n$ (resp. $\mathcal{R}(A, I) = \bigoplus_{n \in \mathbb{Z}} I^n X^n$) is the Rees ring (resp. the extended Rees ring) of the ideal I and where $u = X^{-1}$.

The concept of analytic spread for an ideal has been extended in various ways to filtrations, see [DDS] and [O]. In particular it was shown in [DDS] that for a given filtration $f = (I_n)$ on a noetherian local ring (A, m, k), the numerical function $n \longmapsto \varphi_f(n) = dim_k(\frac{I_n}{m\ I_n})$ need not be of polynomial type even for nice filtrations like noetherian filtrations. But as far as we know, investigations on the asymptotic behaviour of this function go back to [DS2], where it was shown that φ_f is a quasi-polynomial function if f is noetherian.

In the present paper, we investigate the asymptotic behaviour of the numrical function φ_f when f is $I-$ good or strongly noetherian.

In section 2 we recall some classical types of filtrations and some graded rings associated to a filtration which are used in the sequel of this paper.

In section 3, we investigate the asymptotic behaviour of the numerical function φ_f when f is an $I-$ good filtration on a noetherian local ring (A, m, k). We show in particular that for such a type of filtrations , the associated numerical function φ_f is of polynomial type and that the analytic spread of f is equal to the analytic spread of I, so it does not depend on the $I-$good filtration f.

In section 4 it is shown that if $f = (I_n)$ is a strongly noetherian filtration then the numerical function $n \longmapsto \varphi_f(n) = dim_k(\frac{I_n}{m\ I_n})$ is of polynomial type and that its degree is equal to

2 Graded rings associated to filtrations

2.1

By a filtration on the commutative ring A we mean a family $f = (I_n)_{n\in\mathbb{Z}}$ of ideals of A such that $I_0 = A, I_{n+1} \subseteq I_n \forall n$ and $I_m I_n \subseteq I_{m+n} \forall m, n \in \mathbb{Z}$.

For any ideal I of A, the filtration $f_I = (I^n)$, where $I^n = 0$ for all $n \leq 0$, is called the $I-$adic filtration.

The filtration $f = (I_n)_{n\in\mathbb{Z}}$ is said to be $I-$good, where I is an ideal of A, if $I\ I_n \subseteq I_{n+1}$ and if $I\ I_n = I_{n+1} \forall n \gg 0$ which means for all large n.

Similarly to ideals, one can associate to any filtration $f = (I_n)$ of the ring A the following graded rings :

- the fiber cone of f which is the ring $G(A, m, f) = \bigoplus_{n\geq 0} \dfrac{I_n}{m\ I_n}$
- the Rees ring of f which is the ring $R(A, f) = \bigoplus_{n\geq 0} I_n X^n$ and
- the extended Rees ring of f which is the ring $\mathcal{R}(A, f) = \bigoplus_{n\in\mathbb{Z}} I_n X^n$.

In particular if $f = f_I = (I^n)$ is the I-adic filtration, then we set $G(A, m, f_I) = G(A, m, I), R(A, f_I) = R(A, I)$ and $\mathcal{R}(A, f_I) = \mathcal{R}(A, I)$

It should be recalled here that the multiplication of the ring $G(A, m, f)$ is defined by linearity from the formula $\forall p, q \in \mathbb{N}, \forall a_p \in I_p, \forall b_q \in I_q, (a_p\ + mI_q)(b_q + mI\ _q) = a_p b_q + mI\ _{p+q}$.

Remark 2

The ideal $m\ R(A, f) = \bigoplus_{n\geq 0} mI_n X^n$ of $R(A, f)$ is a graded.
So $\dfrac{R(A,f)}{m\ R(A,f)}$ is graded by the $\dfrac{I_n X^n + m\ R(A,f)}{m\ R(A,f)} \backsimeq \dfrac{I_n X^n}{I_n X^n \cap m\ R(A,f)} \backsimeq \dfrac{I_n}{mI_n} \forall n \geq 0$.

2.2

The concept of analytic spread for an ideal has been extended in various ways to filtrations, see [DDS] and [O]. In particular it was shown in [DDS] that for a given filtration $f = (I_n)$, the numerical function $n \longmapsto \varphi_f(n) = dim_k(\dfrac{I_n}{mI_n})$ is not necessarily of polynomial type even for nice filtrations like noetherian filtrations. But actually, questions about the asymptotic nature of this fuction were raised for the first time in [DS2] where the authors had used the concept of quasi-polynomial function to define a degree for the function φ_f . Here we will show that if f is an I-good filtration on a noetherian local ring (A, m), then the function φ_f is of polynomial type of degree dim $\dfrac{R(A,f)}{m\ R(A,f)} - 1$.

3 The asymptotic behaviour of φ_f for $I-$ good filtrations

We will start by recalling the modern version of the well known Hilbert Theorem .

Theorem 3 ***Hilbert Theorem.*** *Let $A = \bigoplus_{n\geq 0} A_n$ be a positively graded noetherian ring. Assume that A is of the form $A = A_0[x_1, ..., x_r]$, where each $x\ _i$ is homogeneous of degree 1 and that $A\ _0$ is an artinian local ring . Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated positively graded A-module of dimension d. Then the Hilbert function $H(M, -)$ (resp. the cummulative Hilbert function $H*(M, -)$) is of polynomial type of degree d - 1 (resp. d)*

We recall that under the notations and hypotheses of Theorem 3, the numerical function $H(M, -) : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined as $H(M, n) = \ell_{A_0}(M_n) \forall n \in \mathbb{Z}$, where $\ell_{A_0}(M_n)$ is the length of the A_0 -module M_n, is called the Hilbert function of M. The cummulative Hilbert function of M is the function $n \longmapsto H*(n) = \sum_{j\leq n} H(M, j)$.

Lemma 4 *Let $f = (I_n)$ be a filtration on A and let$\theta_f : R(A, f) \longrightarrow G(A; m, f)$ be defined as follows :$\forall$ $z = \sum_{n \geq 0} a_n\ X^n$, where $a_n \in I_n\ \forall\ n$, $\theta_f\ (\ z\ ) = \sum_{n \geq 0}(a_n\ + m\ I_n)$. Then θ_f is a surjective morphism of graded rings which induces an isomorphism of graded rings $\overline{\theta_f}\ :\ \dfrac{R(A,\ f)}{m\ R(A,\ f)} \longrightarrow G(A, m, f)$.*

Proof. . This follows immediately from Remark 2. We have ker θ = m $R(A, f)$. So by the isomorphism Theorem the isomorphism $\overline{\theta_f}$ is defined as $\overline{\theta_f}\ (z + mR(A, f)\ ) = \theta_f\ (z)\ \forall$ z $\in R(A, f)$. It is clear that $\overline{\theta_f}$ is graded of degree 0. ■

Remark 5

Let I be an ideal of A and let f = f_I be the I-adic filtration. Put $\theta_I = \theta_f$ and $\overline{\theta_I} = \overline{\theta_f}$. Then it follows from Lemma 4 that $\overline{\theta_I}\ :\ \dfrac{R(A,\ I)}{m\ R(A,\ I)} \longrightarrow$ G (A ; m, I) is an isomorphism of graded rings.

Suppose that I $\subseteq I_1$. Then $G(A, m, f)$ is endowed with a structure of $G(A, m, I)$ - module as follows : $\forall p, q \in \mathbb{N}$, $\forall a_p \in I^p$, $\forall b_q \in I_q$, $(a_p + mI^p)(b_q + mI_q) = a_p b_q + mI_{p+q}$.
It is easily checked that this multiplication is independent of the choice of representatives of $\dfrac{I^p}{mI^p}$ and $\dfrac{I_q}{mI_q}$. It is then extended by linearity to $G(A, m, I) \times G(A, m, f)$. Similarly $R(A, f)$ is obviously a graded $R(A, I)$ - module and $\dfrac{R(A, f)}{mR(A, f)}$ is a graded $\dfrac{R(A, I)}{mR(A, I)}$ - module. We have :
$\forall \alpha \in R(A, I)$ and $\forall z \in R(A, f)$, $\theta_f(\alpha z) = \theta_I(\alpha)\theta_f(z)$, hence
$\overline{\theta_f}(\alpha + mR(A, I)(z + mR(A, f)) = \overline{\theta_f}(\alpha z + mR(A, f)) = \overline{\theta_I}(\alpha + mR(A, I))\overline{\theta_f}(z + mR(A, f))$

Lemma 6 . *Let $f = (I)_n$ be an I-good filtration on the noetherian local ring (A, m). Then the Annihilator of the $\dfrac{R(A, I)}{mR(A, I)}$ - module $\dfrac{R(A, f)}{mR(A, f)}$ is equal to $\dfrac{R(A, I) \cap mR(A, f)}{mR(A, I)}$ and the Krull dimension of the $\dfrac{R(A, I)}{mR(A, I)}$ - module $\dfrac{R(A, f)}{mR(A, f)}$ is equal to the Krull dimension $\gamma_m(f)$ of the ring $\dfrac{R(A, f)}{mR(A, f)}$ which is as well the analytic spread $\lambda(I)$ of the ideal I.*

Proof. . For the definition of the structure of $\dfrac{R(A,\ I)}{m\ R(A,\ I)}$ - module on $\dfrac{R(A,\ f)}{m\ R(A,\ f)}$, see Remark 2.2. It is easily checked that Ann $_{B(I)}$ $\left(\dfrac{R(A,\ f)}{m\ R(A,\ f)} \right) = \dfrac{R(A,\ I) \cap m\ R(A,\ f)}{m\ R(A,\ I)}$, where B (I) = $\dfrac{R(A,\ I)}{m\ R(A,\ I)}$. On the other hand ■

$R(A,\ f)$ is integral over $R(A,\ I)$. Hence $\dfrac{R(A,\ f)}{m\ R(A,\ f)}$ is integral over $\dfrac{R(A,\ I)}{R(A,\ I) \cap m\ R(A,\ f)}$.

Therefore the Krull dimension γ_m (f) of the ring $\dfrac{R(A,\ f)}{m\ R(A,\ f)}$ is

γ_m (f) = dim $\dfrac{R(A,\ I)}{R(A,\ I) \cap m\ R(A,\ f)}$. But if we put B (I) = $\dfrac{R(A,\ I)}{m\ R(A,\ I)}$, then the Krull dimension of the

B (I) - module $\dfrac{R(A,\ f)}{m\ R(A,\ f)}$ is dim $\left(\dfrac{B(I)}{Ann_{B(I)}\ \dfrac{R(A,\ f)}{m\ R(A,\ f)}} \right)$. The rings $\dfrac{B(I)}{Ann_{B(I)}\ \dfrac{R(A,\ f)}{m\ R(A,\ f)}}$ and $\dfrac{R(A,\ I)}{R(A,\ I) \cap m\ R(A,\ f)}$ are isomorphic . So they have the same dimension which is γ_m (f) . The last part follows from the equalities $\gamma_m(f) = \gamma_m(f_I) = \lambda(I)$ shown by [DS] , Theorem 5.7

Lemma 7 . *Let f = (I_n) be an I-good filtration on the noetherian local ring (A, m). Then the graded ring G(A, m, f) is a finitely generated G (A, m, I) - module.*

Proof. There exists an integer n_0 such that $\forall$ n $\geq$ 0, $I_{n_0+n} = I^n I_{n_0}$. Therefore ■
$\frac{I_{n_0+n}}{m I_{n_0+n}} = \frac{I^n}{m I^n} \frac{I_{n_0}}{m I_{n_0}}$. So G(A, m, f) is genrated as G (A, m, I) -module by $\sum_{n=0}^{n=n_0} \frac{I_n}{m I_n}$.
The conclusion follows since each ideal I_n is finitely generated.

Theorem 8 *Let (A, m, k) be a noetherian local ring, I an ideal of A, f = (I_n) an I-good filtration on A. Then the numerical function* $n \longmapsto \varphi_f(n) = \dim_k (\frac{I_n}{m I_n})$ *is of polynomial type.*

Furthermore $\deg \varphi_f = \dim \frac{R(A,\ f)}{m\ R(A,\ f)} - 1 = \lambda (I) - 1 = \deg \varphi_I$

Proof. G (A, m, I) is a noetherian graded ring of the form G (A, m, I) = k [$t_1, \dots, t_r$], where each t_i is homogeneous of degree 1. By Lemma 2.5, G (A, m, f) is a finitely genrerated G (A, m, I) - module. Hence by Hilbert's Theorem 2.1 the numerical function $n \longmapsto \varphi_f(n) = \dim_k (\frac{I_n}{m I_n})$ is of polynomial type of degree d-1 where d denotes the Krull dimension of the G (A, m, I) - module G (A, m, f) . It can be deduced from Remark 2.3 that d is equal to the Krull dimension of the $\frac{R(A,\ I)}{m\ R(A,\ I)}$ - module $\frac{R(A,\ f)}{m\ R(A,\ f)}$ which is equal to γ_m (f) $= \dim \frac{R(A,\ f)}{m\ R(A,\ f)}$ by Lemma 2.4. The remaining equalities follow also from Lemma 2.4. ■

4 The asymptotic behaviour of φ_f for a strongly noetherian filtration f

Definition 9

Let k and d be integers with k $\geq$ 1 and d $\geq$ -1. A numerical function $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is called a quasi-polynomial function of period k and degree d if there exists a sequence F = ($F_0, F_1, \dots, F_{k-1}$) of polynomials $F_j \in \mathbb{Q}$ [X] such that for j = 0, 1, ..., k-1,

(i) φ (n) = F_j (n) $\forall$ n $\gg$ 0 with $n \equiv j (mod k)$ and

(ii) $\mathrm{Max}_{0 \leq j \leq k-1} \deg F_j = d$.

F is called the quasi-polynomial associated with φ and d is called the degree of F.

Here we need the following theorem which is part of Theorem 2.7 of [DS].

Theorem 10 *Let* $A = \bigoplus_{n \geq 0} A_n$ *be a positively graded noetherian ring of finite Krull dimension which is of the form* $A = A_0[x_1, \dots, x_r]$, *where each* x_i *is homogeneous of degree* $k_i \geq 1$. *Assume that* A_0 *is an artinian ring . Let* $M = \bigoplus_{n \in \mathbb{Z}} M_n$ *be a finitely generated positively graded A-module of Krull dimension d. Let* $k = LCM(k_1, k_2, \dots, k_r)$, *then :*

1) *The Hilbert function* $H(M, -)$ *of M is a quasi-polynomial function of period k and degree* $d-1$ *and if* $F = (F_0, F_1, \dots, F_{k-1})$ is the associated quasi-plynomial, then $degF = d - 1$.

2) *The cummulative Hilbert function* $H*(M, -)$ *is of polynomial type of degree d . In addition, if* $G = (G_0, G_1, \dots, G_{k-1})$ *denotes the quasi-polynomial associated with the function* $H*(M, -)$, *then all the polynomials* G_j *have the same degree which is equal to d and the same leading coefficient.*

A filtration $f = (I_n)$ on the ring A is called strongly noetherian if there exists an integer N $\geq$ 1 such that $\forall m, n \in \mathbb{N}$ with $m \geq N$ and $n \geq N$, it holds $I_{m+n} = I_m I_n$. One deduces that $I_{mn} = I_m^n \forall m \geq N$ and $\forall n \geq 0$, which means that if we denote by $f^{(m)}$ the filtration $(I_{mn})_{n \geq 0}$, then $f^{(m)} = f_{I_m}$ is the I_m -adic filtration for all m $\geq N$.

Theorem 11 *Let* (A, m, k) *be a noetherian local ring,* $f = (I_n)$ *a strongly noetherian filtration on A. Then the numerical function* $n \longmapsto \varphi_f(n) = \dim_k(\frac{I_n}{mI_n})$ *is of polynomial type of degree* $\dim \frac{R(A, f)}{mR(A, f)}$ *-1 .*

Proof. Let N be an integer such that $I_{m+n} = I_m I_n$ $\forall$ m ,n $\in \mathbb{N}$ with m $\geq N$ and $n \geq N$. In particular for all m $\geq N$,

$f^{(m)} = (I_{mn})_{n \geq 0} = (I_m{}^n)_{n \geq 0} = f_{I_m}$. Then following Theorem 8, for m $\gg 0$ the numerical function

$\varphi_{f(m)}$ is of polynomial type since $f^{(m)}$ is I_m -adic. Let P_m be the polynomial associated to $\varphi_{f(m)}$. We have deg $P_m = dim \dfrac{R(A, f^{(m)})}{m R(A, f^{(m)})}$ -1 $= dim \dfrac{R(A, f)}{m R(A, f)}$ -1 by [DDS], Proposition 2.4. Hence deg P_m does not depend on m for m $\gg 0$. Set d = deg P_m $\forall$ m $\geq N$. Then P_m is of the form $P_m = \sum_{i=0}^{i=d} a_{m,i} X^i$, where $a_{m,i} \in \mathbb{Q}$ $\forall$ m, i. So $\forall$ r $\gg 0$ and $\forall$ s $\gg 0$, we have $P_m(rs) = \sum_{i=0}^{i=d} a_{m,i}(rs)^i = \varphi_{f(m)}$ (r s) $= dim_k \left(\dfrac{I_{mrs}}{m I_{mrs}}\right) =$

$\varphi_{f(r)}$ (m s) $= P_r(ms) = \sum_{i=0}^{i=d} a_{r,i}(ms)^i$. It follows that $\sum_{i=0}^{i=d}(a_{m,i} r^i) s^i = \sum_{i=0}^{i=d} (a_{r,i} m^i) s^i$ for all s $\gg 0$ hence $a_{m,i} r^i = a_{r,i} m^i$ for i = 0, 1, ..., d and $\dfrac{a_{m,i}}{m^i} = \dfrac{a_{r,i}}{r^i} = a_i$ for i = 0, 1, ..., d. Indeed this ratio depends only on i and not on m nor r. Let P be the polynomial $P(X) = \sum_{i=0}^{i=d} a_i X^i$. Then

$P_m(X) = \sum_{i=0}^{i=d} a_i m^i X^i = P(mX)$. So for $m, r \gg 0$, $\varphi_f(mr) = dim_k \left(\dfrac{I_{mr}}{m I_{mr}}\right) = \varphi_{f(m)}$ (r) $= P_m(r) = \sum_{i=0}^{i=d} a_i m^i r^i = \sum_{i=0}^{i=d} a_i (mr)^i = P(mr)$. To be more precise, by Theorem 4.3 of [DS2], φ_f is a quasi-polynomial function since f is strongly noetherian hence noetherian. Let

$F = (F_0, F_1, ..., F_{k-1})$ be the quasi - polynomial associated with φ_f where k is the period of φ_f. Then

$\varphi_f(mk+j) = F_j(mk+j) \forall m \gg 0$ and for $j = 0, 1, ..., k-1$. Now if m and s are $\gg 0$, then

$P((sk+j)(mk+1) = \varphi_f((sk+j)(mk+1)) = \varphi_f((smk^2+sk+jmk+j)) = \varphi_f((smk+s+jm)k+j)) = F_j((smk+s+jm)k+j) = F_j((sk+j)(mk+1)$. The two polynomials P and F_j coincide on an infinite set of integers. So $P = F_0 = F_1 = ... = F_{k-1}$ and φ_f is of plynomial type, q.e.d. ■

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